

AN EXTENSION OF THE CHEEGER-MÜLLER THEOREM FOR A CONE

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ABSTRACT. We prove the equality of the analytic torsion and the intersection R torsion of the cone over an odd dimensional compact manifold.

1. INTRODUCTION

The classical Cheeger-Müller theorem proves equality between analytic torsion and Reidemeister R torsion for closed Riemannian manifolds [22] [3] [20]. When the manifold has a boundary, a boundary term appears. This boundary term (given by Lück in [17] when the metric is a product near the boundary) has been explicitly given in the general case in some recent works of Brüning and Ma [1] [2]. For a compact connected oriented Riemannian manifold (W, g) with boundary the Cheeger Müller theorem reads (a similar formula is valid for the relative case)

$$\log T_{\text{abs}}((W, g); \rho) = \log \tau_{\text{R}}((W, g); \rho) + \frac{1}{2} \chi(\partial W) \log 2 + A_{\text{BM,abs}}(\partial W),$$

where $T_{\text{abs}}((W, g); \rho)$ and $\tau_{\text{R}}((W, g); \rho)$ are the analytic torsion with absolute BC on the boundary, and the Reidemeister R torsion of (W, g) , with respect to the rank one orthogonal representation ρ of the fundamental group of W and with the basis for homology fixed as in [22] (see Sections 2.2 and 3.4 for details), respectively, χ is the Euler characteristic, and A_{BM} is the anomaly boundary term of Brüning and Ma, see [1] and [2] for absolute and relative BC, respectively.

In this work we prove the following extension of the Cheeger Müller theorem, where CW is the cone over W (see Section 2.4 for details) and where the usual R torsion is replaced by the intersection torsion $I\tau$ defined by A. Dar in [6] (see Section 3.4 for details). The proof follows at once from the formula for the analytic torsion given in Theorem 2.1 in Section 2.4, and the duality formula for the intersection torsion proved in Proposition 4.1 in the last section. Since the metric on CW is fixed by the metric of W , and the unique representation of the fundamental group is the trivial one, both these quantities will be omitted in the notation.

Theorem 1.1. *Let (W, g) be a compact connected oriented Riemannian odd dimensional manifold without boundary. Then,*

$$\begin{aligned} \log T_{\text{abs}}(CW) &= \log I\tau_{\text{R}}(CW) + A_{\text{BM,abs}}(W), \\ \log T_{\text{rel}}(CW) &= \log I\tau_{\text{R}}(CW, \partial CW) + A_{\text{BM,rel}}(W). \end{aligned}$$

If the dimension of W is even, results for the analytic torsion (see [14]) exist and results for the duality of the intersection R torsion are given in the last section of this work. However, as observed in [14], some extra term in the analytic torsion appears, and the extension of the Cheeger Müller

theorem depends upon a geometric interpretation of such term. Since this is not clear yet, we prefer to omit not illuminating formulas.

We conclude with two remarks. First, we observe that the result of this work will give the extension of the Cheeger Müller theorem for a general space with conical singularities (as define in [4]) once it will be available a suitable gluing theorem, extending the one a for product metric near the boundary, proved by S.M. Vishik in [29]. Such a theorem is probably contained in the work to appear of Brüning and Ma [2]. Second, since in general pseudomanifold are modeled on cones, a generalization of the result of this work for a cone over a pseudomanifold could be used to obtain a generalization of the Cheeger Müller theorem for pseudomanifolds. There are works in progress in these directions.

2. ANALYTIC TORSION

This section is essentially based on [13] and [14], and we refer to those papers for further details.

2.1. Geometric setting. Let (W, g) be a compact connected oriented Riemannian manifold of dimension m with boundary ∂W and Riemannian structure g . Let $\rho : \pi_1(W) \rightarrow O(k, \mathbb{R})$ be a representation of the fundamental group of W , and let E_ρ be the associated vector bundle over W with fibre \mathbb{R}^k and group $O(k, \mathbb{R})$, $E_\rho = \widetilde{W} \times_\rho \mathbb{R}^k$. Let $\Omega(W, E_\rho)$ denote the graded linear space of smooth forms on W with values in E_ρ . The exterior differential on W defines the exterior differential on $\Omega^q(W, E_\rho)$, $d : \Omega^q(W, E_\rho) \rightarrow \Omega^{q+1}(W, E_\rho)$. The metric g defines an Hodge operator on W and hence on $\Omega^q(W, E_\rho)$, $\star : \Omega^q(W, E_\rho) \rightarrow \Omega^{m-q}(W, E_\rho)$, and, using the inner product $\langle \cdot, \cdot \rangle$ in \mathbb{R}^k , an inner product on $\Omega^q(W, E_\rho)$ is defined by

$$(\omega, \eta) = \int_W \langle \omega \wedge \star \eta \rangle.$$

Near the boundary there is a natural splitting of ΛW as direct sum of vector bundles $\Lambda T^* \partial W \oplus N^* W$, where $N^* W$ is the dual to the normal bundle to the boundary, and the smooth forms on W near the boundary decompose as $\omega = \omega_{\text{tan}} + \omega_{\text{norm}}$, where ω_{norm} is the orthogonal projection on the subspace generated by dx , the one form corresponding to the outward pointing unit normal vector to the boundary, and ω_{tan} is in $C^\infty(W) \otimes \Lambda(\partial W)$. We write $\omega = \omega_1 + dx \wedge \omega_2$, where $\omega_j \in C^\infty(W) \otimes \Lambda(\partial W)$, and

$$\star \omega_2 = -dx \wedge \star \omega.$$

Define absolute and relative boundary conditions by

$$B_{\text{abs}}(\omega) = \omega_{\text{norm}}|_{\partial W} = \omega_2|_{\partial W} = 0, \quad B_{\text{rel}}(\omega) = \omega_{\text{tan}}|_{\partial W} = \omega_1|_{\partial W} = 0.$$

Let $\mathcal{B}(\omega) = B(\omega) \oplus B((d + d^\dagger)(\omega))$. The adjoint d^\dagger and the Laplacian $\Delta = (d + d^\dagger)^2$ operators are defined on the space of sections with values in E_ρ , the Laplacian with boundary conditions $\mathcal{B}(\omega) = 0$ is self adjoint, and the spaces of the harmonic forms with boundary conditions are

$$\begin{aligned} \mathcal{H}^q(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)} \omega = 0\}, \\ \mathcal{H}_{\text{abs}}^q(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)} \omega = 0, B_{\text{abs}}(\omega) = 0\}, \\ \mathcal{H}_{\text{rel}}^q(W, E_\rho) &= \{\omega \in \Omega^q(W, E_\rho) \mid \Delta^{(q)} \omega = 0, B_{\text{rel}}(\omega) = 0\}. \end{aligned}$$

2.2. De Rham maps. Let K be a cellular or simplicial decomposition of W and L of ∂W . Then we have the following de Rham maps \mathcal{A}^q (that induce isomorphisms in cohomology),

$$\mathcal{A}^q : \mathcal{H}_{\text{abs}}^q(W, E_\rho) \rightarrow C^q(W; E_\rho), \quad \mathcal{A}_{\text{rel}}^q : \mathcal{H}_{\text{rel}}^q(W, E_\rho) \rightarrow C^q((W, \partial W); E_\rho),$$

with

$$\mathcal{A}^q(\omega)(c \otimes_\rho v) = \mathcal{A}_{\text{rel}}^q(\omega)(c \otimes_\rho v) = \int_c (\omega, v),$$

where $c \otimes_\rho v$ belongs to $C^q(W; E_\rho)$ in the first case, and belongs to $C^q((W, \partial W); E_\rho)$ in the second case, and c is identified with the q -subcomplex (simplicial or cellular) that c represents. Following Ray and Singer, we introduce the de Rham maps \mathcal{A}_q :

$$\begin{aligned} \mathcal{A}_q^{\text{rel}} : \mathcal{H}_{\text{rel}}^q(W, E_\rho) &\rightarrow C_q((W, \partial W); E_\rho), & \mathcal{A}_q^{\text{rel}} : \omega &\mapsto (-1)^{(m-1)q} \mathcal{P}_q^{-1} \mathcal{A}_{\text{abs}}^{m-q} \star (\omega), \\ \mathcal{A}_q^{\text{abs}} : \mathcal{H}_{\text{abs}}^q(W, E_\rho) &\rightarrow C_q(W; E_\rho), & \mathcal{A}_q^{\text{abs}} : \omega &\mapsto (-1)^{(m-1)q} \mathcal{P}_q^{-1} \mathcal{A}_{\text{rel}}^{m-q} \star (\omega), \end{aligned}$$

both defined by

$$(2.1) \quad \mathcal{A}_q^{\text{abs}}(\omega) = \mathcal{A}_q^{\text{rel}}(\omega) = (-1)^{(m-1)q} \sum_{j,i} \left(\int_{\hat{c}_{q,j}} (\star \omega, e_i) \right) c_{q,j} \otimes_\rho e_i,$$

where the sum runs over all q -simplices $c_{q,j}$ of W in the first case, but runs over all q -simplices $c_{q,j}$ of $W - \partial W$ in the second case. Here $\mathcal{P}_q : C_q(K, L; \mathbb{Z}) \rightarrow C^{m-q}(\hat{K} - \hat{L}; \mathbb{Z})$ is the Poincaré map, and \hat{c} denotes the dual block cell of c .

2.3. Zeta function and analytic torsion. The Laplace operator on forms $\Delta^{(q)}$, with boundary conditions $\mathcal{B}_{\text{abs/rel}}$, has a pure point spectrum $\text{Sp} \Delta_{\text{abs/rel}}^{(q)}$ consisting of real non negative eigenvalues. The sequence $\text{Sp}_+ \Delta_{\text{abs/rel}}^{(q)}$ is a totally regular sequence of spectral type accordingly to [13] Section 4, and the *forms valued zeta function* is the associated zeta function, defined by

$$\zeta(s, \Delta_{\text{abs/rel}}^{(q)}) = \zeta(s, \text{Sp}_+ \Delta_{\text{abs/rel}}^{(q)}) = \sum_{\lambda \in \text{Sp}_+ \Delta_{\text{abs/rel}}^{(q)}} \lambda^{-s},$$

when $\text{Re}(s) > \frac{m}{2}$, and by analytic continuation elsewhere. The *analytic torsion* $T_{\text{abs/rel}}((W, g); \rho)$ of (W, g) with respect to the representation ρ is defined by

$$\log T_{\text{abs/rel}}((W, g); \rho) = \frac{1}{2} \sum_{q=1}^m (-1)^q q \zeta'(0, \Delta_{\text{abs/rel}}^{(q)}).$$

2.4. The analytic torsion of a cone. Let (W, g) be an orientable compact connected Riemannian manifold of dimension m without boundary and with Riemannian structure g . We denote by $C_l W$ the space $([0, l] \times W) / (\{0\} \times W) = (0, l] \times W \cup \{p\}$, where p is the vertex of the cone, i.e. the image of $\{0\} \times W$ under the quotient map, with the metric

$$g_C = dx \otimes dx + x^2 g,$$

on $(0, l] \times W$, and we call it the *finite metric cone* over W (see [14] 3.1 for details). The analytic torsion of a cone over a sphere (i.e. $W = S^m$) was studied in [13]. The result is based on one side on works of J. Cheeger on the Hodge theory of L^2 forms [4] [5] [21], and on the other on works of M. Spreafico on zeta invariants for double sequences [24] [25] [26] [27]. In the general case, extending the approach used for the spheres in [13], we have the following result:

Theorem 2.1. *The analytic torsion of the cone $C_l W$ on an orientable compact connected Riemannian manifold (W, g) of odd dimension $m = 2p - 1$ is*

$$\log T_{\text{abs}}(C_l W, g_C) = \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} \text{rk} H_q(W; \mathbb{Q}) \log \frac{2(p-q)}{l} + \frac{1}{2} \log T(W, l^2 g) + S(\partial C_l W),$$

where the singular term $S(\partial C_l W)$ only depends on the boundary of the cone:

$$S(\partial C_l W) = \frac{1}{2} \sum_{q=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^j \text{Res}_0 \Phi_{2k+1,q}(s) \binom{-\frac{1}{2}-k}{j-k} \sum_{h=0}^q (-1)^h \text{Res}_1 \zeta \left(s, \tilde{\Delta}^{(h)} \right) (q-p+1)^{2(j-k)},$$

(the functions $\Phi_{2k+1,q}(s)$ are some universal functions, explicitly known by some recursive relations, and $\tilde{\Delta}$ is the Laplace operator on forms on the section of the cone) and coincides with the anomaly boundary term of Brüning and Ma, namely $S(\partial C_l W) = A_{\text{BM,abs}}(\partial C_l W)$.

The proof of Theorem 2.1 is based on analytic tools and is essentially the same as the proof of similar results for the spheres given in [13]. In the general case treated here, using the same method and a similar strategy, we just need to solve several technical problems, that can be quite hard, and require long difficult analysis. Thus, we present here a condensed version of the proof to make this work self consistent, all details will appear somewhere else (see also the online preprint [14]). The proof consists of the following six steps.

2.4.1. The spectrum. Using the explicit form for the solutions of the eigenvalues equation for the Laplace operator on forms Δ on the cone $C_\infty W$ given in [4] (see also [21]), and applying BC at $x = l$, we prove that Δ has a semi bounded self adjoint extension with pure point spectrum [14] 4.3:

$$\begin{aligned} \text{Sp}_+ \Delta_{\text{abs}}^{(q)} = & \left\{ m_{\text{cex},q,n} : \hat{j}_{\mu_{q,n},\alpha_q,k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},\alpha_{q-1},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{cex},q-1,n} : \hat{j}_{\mu_{q-1,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \cup \left\{ m_{q-2,n} : \hat{j}_{\mu_{q-2,n},k}^2 / l^2 \right\}_{n,k=1}^\infty \\ & \cup \left\{ m_{\text{har},q,0} : \hat{j}_{|\alpha_q|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty \cup \left\{ m_{\text{har},q-1,0} : \hat{j}_{|\alpha_{q-1}|,\alpha_q,k}^2 / l^2 \right\}_{k=1}^\infty, \end{aligned}$$

where the $\hat{j}_{\mu,k}$ are the zeros of the Bessel function $J_\mu(x)$, the $\hat{j}_{\mu,c,k}$ are the zeros of the function $\hat{J}_{\mu,c}(x) = cJ_\mu(x) + xJ'_\mu(x)$, $c \in \mathbb{R}$,

$$\alpha_q = \frac{1}{2}(1 + 2q - m), \quad \mu_{q,n} = \sqrt{\lambda_{q,n} + \alpha_q^2},$$

and $m_{\text{har},q,n}$, $m_{\text{cex},q,n}$ denote the dimensions of the eigenspaces of an orthonormal base of $\Gamma(W, \Lambda^{(q)} T^* W)$ consisting of harmonic, coexact and exact eigenforms of the restriction of the Laplace operator $\tilde{\Delta}$ on W , associated to the eigenvalue $\lambda_{q,n}$. A similar result holds for relative BC.

2.4.2. The torsion zeta function. We define the *torsion zeta function* by

$$t_{\text{abs/rel}}(s) = \frac{1}{2} \sum_{q=1}^{m+1} (-1)^q q \zeta(s, \Delta_{\text{abs/rel}}^{(q)}),$$

so that $\log T_{\text{abs/rel}}(C_l W) = t'_{\text{abs/rel}}(0)$. The first result is Poincaré duality for the analytic torsion of a cone [14] 5.1:

$$\log T_{\text{abs}}(C_l W) = (-1)^m \log T_{\text{rel}}(C_l W).$$

Next, we show that the torsion zeta function simplifies as [14] 6.1 (we assume from now on $m = 2p - 1$):

$$\begin{aligned} t(s) = & \frac{l^{2s}}{2} \sum_{q=0}^{p-2} (-1)^q \left(\sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \left(2j_{\mu_q,n,k}^{-2s} - \hat{j}_{\mu_q,n,\alpha_q,k}^{-2s} - \hat{j}_{\mu_q,n,-\alpha_q,k}^{-2s} \right) \right) \\ & + (-1)^{p-1} \frac{l^{2s}}{2} \left(\sum_{n,k=1}^{\infty} m_{\text{cex},p-1,n} \left(j_{\mu_{p-1},n,k}^{-2s} - (j'_{\mu_{p-1},n,k})^{-2s} \right) \right) \\ & - \frac{l^{2s}}{2} \sum_{q=0}^{p-1} (-1)^q \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q}) \sum_{k=1}^{\infty} \left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s} \right). \end{aligned}$$

We set

$$\begin{aligned} (2.2) \quad Z_q(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} j_{\mu_q,n,k}^{-2s}, & \dot{Z}_q(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} (j'_{\mu_q,n,k})^{-2s}, \\ Z_{q,\pm}(s) &= \sum_{n,k=1}^{\infty} m_{\text{cex},q,n} \hat{j}_{\mu_q,n,\pm\alpha_q,k}^{-2s}, & z_q(s) &= \sum_{k=1}^{\infty} \left(j_{-\alpha_{q-1},k}^{-2s} - j_{-\alpha_q,k}^{-2s} \right), \end{aligned}$$

for $0 \leq q \leq p-1$, and

$$\begin{aligned} t_{p-1}(s) &= Z_{p-1}(s) - \dot{Z}_{p-1}(s), \\ t_q(s) &= 2Z_q(s) - Z_{q,+}(s) - Z_{q,-}(s), \quad 0 \leq q \leq p-2. \end{aligned}$$

Then,

$$t(s) = \frac{l^{2s}}{2} \sum_{q=0}^{p-1} (-1)^q t_q(s) - \frac{l^{2s}}{2} \sum_{q=0}^{p-1} (-1)^q r_q z_q(s),$$

and

$$\begin{aligned} (2.3) \quad \log T(C_l W) = t'(0) &= \frac{\log l^2}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z_q(0) + \sum_{q=0}^{p-1} (-1)^q t_q(0) \right) \\ &+ \frac{1}{2} \left(\sum_{q=0}^{p-1} (-1)^{q+1} r_q z'_q(0) + \sum_{q=0}^{p-1} (-1)^q t'_q(0) \right), \end{aligned}$$

where $r_q = \text{rk} \mathcal{H}_q(\partial C_l W; \mathbb{Q})$, and in order to obtain the value of $\log T(C_l W)$ we need to study the derivative at zero of the zeta functions appearing in equation (2.2).

2.4.3. Zeta determinants. We use the method of [24] [27]. In particular we apply Theorems 1 and 2 of [13] and we refer to Section 4 of that paper or to Section 3 of [14] for details. We recall some basic facts for the reader benefit. Let $S = \{a_n\}_{n=1}^{\infty}$ be a sequence of non vanishing complex numbers, ordered by increasing modules, with the unique point of accumulation at infinite, finite exponent of convergence and genus $\mathbf{g}(S)$, and contained in some positive sector of the complex plane. The *zeta function* associated to S is (analytic extension is assumed where necessary)

$$\zeta(s, S) = \sum_{n=1}^{\infty} a_n^{-s},$$

and the *Gamma function* associated to S is

$$\frac{1}{\Gamma(-\lambda, S)} = \prod_{n=1}^{\infty} \left(1 + \frac{-\lambda}{a_n}\right) e^{\sum_{j=1}^{\mathbf{g}(S)} \frac{(-1)^j}{j} \frac{(-\lambda)^j}{a_n^j}}.$$

Next, let $S = \{\lambda_{n,k}\}_{n,k=1}^{\infty}$ be a double sequence of non vanishing complex numbers with unique accumulation point at the infinity, finite exponent and genus. S is said *spectrally decomposable* over some simple sequence $U = \{u_n\}_{n=1}^{\infty}$ if some conditions are satisfied [13] Definition 1. If this is the case, then, by definition, the logarithmic Γ -function associated to S_n/u_n^{κ} has an asymptotic expansion for large n uniformly in λ , of the following type ($\kappa > 0$, and S_n denotes the subsequence with fixed n)

$$\log \Gamma(-\lambda, u_n^{-\kappa} S_n) = \sum_{h=0}^{\ell} \phi_{\sigma_h}(\lambda) u_n^{-\sigma_h} + \sum_{l=0}^L P_{\rho_l}(\lambda) u_n^{-\rho_l} \log u_n + o(u_n^{-r_0}),$$

where σ_h and ρ_l are real numbers with $\sigma_0 < \dots < \sigma_{\ell}$, $\rho_0 < \dots < \rho_L$, the $P_{\rho_l}(\lambda)$ are polynomials in λ satisfying the condition $P_{\rho_l}(0) = 0$, ℓ and L are the larger integers such that $\sigma_{\ell} \leq r_0$ and $\rho_L \leq r_0$. Moreover, for all n , we have the expansions [27] 3.5:

$$\begin{aligned} \log \Gamma(-\lambda, S_n/u_n^{\kappa}) &\sim \sum_{j=0}^{\infty} a_{\alpha_j, 0, n} (-\lambda)^{\alpha_j} + \sum_{k=0}^{\mathbf{g}(S_n/u_n^{\kappa})} a_{k, 1, n} (-\lambda)^k \log(-\lambda), \\ \phi_{\sigma_h}(\lambda) &\sim \sum_{j=0}^{\infty} b_{\sigma_h, \alpha_j, 0} (-\lambda)^{\alpha_j} + \sum_{k=0}^{\mathbf{g}(S_n/u_n^{\kappa})} b_{\sigma_h, k, 1} (-\lambda)^k \log(-\lambda), \end{aligned}$$

for large λ . Setting $(\Lambda_{\theta, c} = \{z \in \mathbb{C} \mid |\arg(z - c)| = \frac{\theta}{2}\})$, oriented counter clockwise):

$$\begin{aligned} \Phi_{\sigma_h}(s) &= \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda_{\theta, c}} \frac{e^{-\lambda t}}{-\lambda} \phi_{\sigma_h}(\lambda) d\lambda dt, \\ (2.4) \quad A_{0,0}(s) &= \sum_{n=1}^{\infty} \left(a_{0,0,n} - \sum_{h=0}^{\ell} b_{\sigma_h, 0, 0} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \\ A_{j,1}(s) &= \sum_{n=1}^{\infty} \left(a_{j,1,n} - \sum_{h=0}^{\ell} b_{\sigma_h, j, 1} u_n^{-\sigma_h} \right) u_n^{-\kappa s}, \quad 0 \leq j \leq p_2, \end{aligned}$$

and assuming that the functions $\Phi_{\sigma_h}(s)$ have at most simple poles at $s = 0$, then ([27] Theorem 3.9, [13] Theorem 3) $\zeta(s, S)$ is regular at $s = 0$, and

$$\begin{aligned} \zeta(0, S) &= -A_{0,1}(0) + \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U), \\ (2.5) \quad \zeta'(0, S) &= -A_{0,0}(0) - A'_{0,1}(0) + \frac{\gamma}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U) \\ &\quad + \frac{1}{\kappa} \sum_{h=0}^{\ell} \operatorname{Res}_0 \Phi_{\sigma_h}(s) \operatorname{Res}_1 \zeta(s, U) + \sum_{h=0}^{\ell} \operatorname{Res}_1 \Phi_{\sigma_h}(s) \operatorname{Res}_0 \zeta(s, U), \end{aligned}$$

where the notation \sum' means that only the terms such that $\zeta(s, U)$ has a pole at $s = \sigma_h$ appear in the sum. Moreover, the result extends in the expected way for a finite linear combination of zeta

functions, with weaker hypotheses on the individual functions (see [13] Corollary 1, [14] Corollary 2.1). It is now clear that we will obtain a formula for the torsion of the cone applying these formulas to the zeta functions defined in equation (2.2). We describe the idea for the function t_q , $0 \leq q \leq p-2$; the other case is similar, up to technical difficulties.

2.4.4. The functions t_q . To study these functions we consider the double sequences $S_q = \{m_{q,n} : j_{\mu_{q,n},k}^2\}_{n,k=1}^\infty$ and $S_{q,\pm} = \{m_{q,n} : j_{\mu_{q,n} \pm \alpha_q, k}^2\}_{n,k=1}^\infty$, since we have that $Z_q(s) = \zeta(s, S_q)$, $Z_{q,\pm}(s) = \zeta(s, S_{q,\pm})$, where $q = 0, 1, \dots, p-2$, $\alpha_q = p - q - 1$. First, we prove [14] Lemma 5.2 that these sequences are spectrally decomposable with respect to the sequence $U_q = \{\mu_{q,n} : \mu_{q,n}\}_{n=1}^\infty$, that is a totally regular sequence of spectral type with infinite order, $\mathbf{g}(U_q) = 2p-1$, and satisfies

$$\zeta(s, U_q) = \zeta_{\text{ce}}\left(\frac{s}{2}, \tilde{\Delta}^{(q)} + \alpha_q^2\right).$$

In particular this shows that the possible poles of $\zeta(s, U_q)$ are at $s = 2p-1-h$, $h = 0, 2, 4, \dots$, and the residues are completely determined by the residues of the function $\zeta_{\text{ce}}(s, \tilde{\Delta}^{(q)})$. Second, we introduce the functions

$$\hat{J}_{\nu,c}(z) = cJ_\nu(z) + zJ'_\nu(z).$$

Recalling the series definition of the Bessel function, and the Hadamard factorization theorem, we have a product expansion for $\hat{J}_{\nu,c}(z)$, and setting [13]

$$\hat{I}_{\nu,c}(z) = e^{-\frac{\pi}{2}i\nu} \hat{J}_{\nu,c}(iz),$$

we have

$$\hat{I}_{\nu, \pm \alpha_q}(z) = \pm \alpha_q I_\nu(z) + z I'_\nu(z) = \left(1 \pm \frac{\alpha_q}{\nu}\right) \frac{z^\nu}{2^\nu \Gamma(\nu)} \prod_{k=1}^\infty \left(1 + \frac{z^2}{\hat{j}_{\nu, \pm \alpha_q, k}^2}\right).$$

This formula provides the following representation for the logarithmic Gamma function associated to the sequence $S_{q,\pm,n}$

$$(2.6) \quad \begin{aligned} & \log \Gamma(-\lambda, S_{q,\pm,n}) \\ &= -\log \hat{I}_{\mu_{q,n}, \pm \alpha_q}(\sqrt{-\lambda}) + \mu_{q,n} \log \sqrt{-\lambda} - \mu_{q,n} \log 2 - \log \Gamma(\mu_{q,n}) + \log \left(1 \pm \frac{\alpha_q}{\mu_{q,n}}\right). \end{aligned}$$

Using the known uniform asymptotic expansions for the Bessel functions and the Gamma function, the formula in equation (2.6) permits to prove that the sequences $S_{q,\pm}$ are spectrally decomposable over the sequence U , and to compute all the relevant parameters. Then, in particular, there exists the following asymptotic expansions for large n , uniform in λ ,

$$\begin{aligned} & 2 \log \Gamma(-\lambda, S_{q,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,+,n}/\mu_{q,n}^2) - \log \Gamma(-\lambda, S_{q,-,n}/\mu_{q,n}^2) \\ &= -2 \log I_{\mu_{q,n}}(\mu_{q,n} \sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n}, \alpha_q}(\mu_{q,n} \sqrt{-\lambda}) + \log \hat{I}_{\mu_{q,n}, -\alpha_q}(\mu_{q,n} \sqrt{-\lambda}) \\ & \quad - 2 \log \mu_{q,n} - \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2}\right) \\ &= \log(1-\lambda) + \sum_{j=1}^{2p-1} \phi_{j,q}(\lambda) \frac{1}{\mu_{q,n}^j} + O\left(\frac{1}{(\mu_{q,n})^{2p}}\right). \end{aligned}$$

The relevant functions in this formula are the function $\phi_{j,q}$. The main information about these functions is: for all j and all $0 \leq q \leq p-2$, $\phi_{j,q}(0) = 0$, and $\text{Res}_{s=0} \Phi_{2j+1,q}(s) = 0$ [14] Lemma

5.6, Corollary 5.1. This information allows to determine the functions $A_{0,0,q}$ and $A_{0,1,q}$ appearing in equation (2.4), we obtain [14] Lemma 6.14: for all $0 \leq q \leq p-2$,

$$\begin{aligned}\mathcal{A}_{0,0,q}(s) &= 2A_{0,0,q}(s) - A_{0,0,q,+}(s) - A_{0,0,q,-}(s) = -\sum_{n=1}^{\infty} \log \left(1 - \frac{\alpha_q^2}{\mu_{q,n}^2} \right) \frac{m_{q,n}}{\mu_{q,n}^{2s}}, \\ \mathcal{A}_{0,1,q}(s) &= 2A_{0,1,q}(s) - A_{0,1,q,+}(s) - A_{0,1,q,-}(s) = \zeta(2s, U_q).\end{aligned}$$

We have now all the ingredients necessary to apply the formulas in equation (2.5). We decompose the result in two parts: the part coming from the sums is called *singular*, the other *regular*. Then, for $0 \leq q \leq p-2$,

$$t_q(0) = t_{q,\text{reg}}(0) + t_{q,\text{sing}}(0), \quad t'_q(0) = t'_{q,\text{reg}}(0) + t'_{q,\text{sing}}(0),$$

where

$$\begin{aligned}t_{q,\text{reg}}(0) &= -\zeta(0, U_q) = -\zeta_{\text{cex}} \left(0, \tilde{\Delta}^{(q)} + \alpha_q^2 \right), & t_{q,\text{sing}}(0) &= 0, \\ t'_{q,\text{reg}}(0) &= -\mathcal{A}_{q,0,0}(0) - \mathcal{A}'_{q,0,1}(0), \\ t'_{q,\text{sing}}(0) &= \frac{1}{2} \sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_1 \zeta(s, U_q) = \frac{1}{2} \sum_{j=0}^{p-1} \text{Res}_0 \Phi_{2j+1,q}(s) \text{Res}_1 \zeta_{\text{cex}} \left(\frac{s}{2}, \tilde{\Delta}^{(q)} + \alpha_q^2 \right).\end{aligned}$$

2.4.5. *The torsion.* An analysis similar to the one of the previous section produces similar results for the function t_{p-1} . Collecting the two sets of results, and studying the zeta functions involved, we have [14] Lemma 7.2

$$\begin{aligned}t_{q,\text{reg}}(0) &= -\zeta_{\text{cex}}(0, \tilde{\Delta}^{(q)}), & 0 \leq q \leq p-2, \\ t'_{q,\text{reg}}(0) &= -\zeta'_{\text{cex}}(0, \tilde{\Delta}^{(q)}), & 0 \leq q \leq p-2, \\ t_{p-1,\text{reg}}(0) &= -\frac{1}{2} \zeta_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}), & t'_{p-1,\text{reg}}(0) = -\frac{1}{2} \zeta'_{\text{cex}}(0, \tilde{\Delta}^{(p-1)}).\end{aligned}$$

Next, recalling equation (2.3), and splitting the regular and the singular part, we see that the singular part is already in the final form of the statement of the Theorem 2.1 [14] Proposition 7.1. For the regular part, some further manipulation is necessary. From one side, using classical zeta function theory (see for example [23]): for all $0 \leq q \leq p-1$,

$$z_q(0) = -\frac{1}{2}, \quad z'_q(0) = \log 2 + \log(p-q).$$

From the other, using Hodge duality we have formula for the torsion where only coclosed eigenvalues appear. This facts produce the formula as stated in Theorem 2.1, and conclude the proof of the first part of the statement [14] Proposition 7.2.

2.4.6. *The boundary term.* To conclude the proof of Theorem 2.1, we need to show that the boundary term $S(\partial C_l W)$ coincides with the anomaly boundary term $A_{\text{abs,BM}}(W)$. In [13] Section 6, it was proved that this is true for W a odd low dimensional sphere. The proof is by direct verification that the two terms coincide. We have a similar proof for any odd dimensional sphere. We also have a direct proof when W is any odd low dimensional manifold. The last proof is based on explicit knowledge of the coefficients in the asymptotic expansion of the heat kernel, given in works of P. Gilkey [8] [14] Section 7. All these proofs are superseded by the following general proof, that

however is based on an indirect argument. Again we refer to [14] Section 8 for details. Consider the conical frustum $C_{[l_1, l_2]}W = [l_1, l_2] \times W$ with basis W ($0 < l_1 < l_2$), and metric

$$g_F = dx \otimes dx + x^2 g.$$

It is clear that $(C_{[l_1, l_2]}W, g_F)$ is a Riemannian manifold with boundary $\partial(C_{[l_1, l_2]}W, g_F) = \partial_1 \sqcup \partial_2$. Using classical duality for analytic and R torsion [14] Section 7, or using the formula for analytic torsion on manifold with boundary and mixed boundary condition proved in [2] (the paper is to appear, we thank the author for making kindly available this result to us), we obtain

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]}W) = 2A_{\text{BM,abs}}(\partial C_l W).$$

On the other side, we compute the analytic torsion by the same means used in the previous sections to compute the analytic torsion of the cone. Calculations are essentially similar, and solving some technical difficulties, we obtain

$$\log T_{\text{rel } \partial_1, \text{abs } \partial_2}(C_{[l_1, l_2]}W) = 2 \log T_{\text{abs, sing}}(C_l W) = 2S(\partial C_l W),$$

completing the proof of Theorem 2.1.

3. INTERSECTION TORSION

Intersection torsion for pseudomanifolds was introduced in works of A. Dar [6] [7]. In these works the case of pseudomanifolds without boundary is considered, and in general all intersection homology theory is developed for the boundaryless case. Here we need to consider the boundary case, but a particular situation where the boundary is in fact a smooth manifold, disjoint from the singular locus. In this particular case it is easy to rework all definitions and the main results of the boundaryless case, as expect. This is the purpose of this section.

3.1. Pseudomenifolds with smooth boundary. We define pseudomanifolds with smooth boundary adapting the definition of pseudomanifolds of [9] 1.1, [12] 1, [16] 4.1. If X is a topological space, we denote by CX the cone over X . By definition the cone and the open cone over the empty set are a point. A *topological pseudomanifold of dimension 0* is a countable set with the discrete topology. A *topological pseudomanifold of dimension n with smooth boundary* is an Hausdorff paracompact topological space X with a filtration by closed subspaces

$$\emptyset = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{n-3} \subset X_{n-2} = X_{n-1} = \Sigma \subset X_n = X,$$

called *stratification*, such that: (1) $X - \Sigma$ is dense in X ; (2) there exists a closed subspace B of X , with $B \cap \Sigma = \emptyset$, such that $M = X - \Sigma$ is an n -manifold with boundary $\partial M = B$, and for each $j \leq n - 2$, for each point $x \in X_j - X_{j-1}$ there exists a compact topological pseudomanifold L of dimension $n - j - 1$ with filtration

$$\emptyset = L_{-1} \subseteq L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-j-3} = L_{n-j-2} \subset L_{n-j-1} = L,$$

and a neighborhood U_x of x in X with an homeomorphism $\varphi : U_x \rightarrow \mathbb{R}^j \times \mathring{C}L$, which respects the stratifications, namely φ maps homeomorphically $U_x \cap X_{j+k+1}$ onto $\mathbb{R}^j \times \mathring{C}L_k$.

The closed subspace $\Sigma = X_{n-2}$ is called the *singular locus* of X . It is a consequence of the definition that each subspace $X_j - X_{j-1}$ is a manifold of dimension j with boundary, and, by condition (2), the boundary of X is disjoint from the singular locus. When the singular locus has dimension 0, then a stratification of X is

$$\emptyset = X_{-1} \subset \Sigma = X_0 = X_1 = \cdots = X_{n-1} \subset X,$$

and X is called a *space with isolated singularities*. In this work we are mainly concerned with this type of pseudomanifolds. If X is a manifold with boundary, then X is a pseudomanifold with a stratification consisting with only one stratum X . For our purpose it is sufficient to work in the piecewise linear category, as in [9]. A *piecewise linear (pl) space* X is a topological space with a class of locally finite simplicial triangulations $\mathcal{T}(X)$: if $T \in \mathcal{T}$ then any (linear) subdivision of T belongs to $\mathcal{T}(X)$, and if $T_1, T_2 \in \mathcal{T}(X)$, then they have a common subdivision in $\mathcal{T}(X)$. A closed pl-subspace of X is a subspace which is a subcomplex of a suitable admissible triangulation of X . We will identify a triangulation of a space with the associated simplicial complex. A *pl-pseudomanifold X of dimension n with smooth boundary* is a pl-space X of dimension n containing two closed disjoint pl-subspaces ∂X and Σ , with Σ of codimension greater or equal to 2, such that $X - \Sigma$ is an oriented pl-manifold of dimension n dense in X and with smooth boundary ∂X . Equivalently, for an (admissible) triangulation of X , then X is the union of the closed n -simplices and each $(n-1)$ -simplex is face of one or two n -simplices, and ∂X is the subcomplex of the $(n-1)$ -simplices that are faces of just one n -simplex. By the same proof as in [12] Prop. 1.4, any pl-pseudomanifold with smooth boundary admits a pl-stratification: a stratification of X is given by setting $X_k = |T_{(k)}|$, where T is an (admissible) triangulation of X , and this stratification is subordinate to the triangulation, meaning that the strata are subcomplexes.

From now on we assume pseudomanifolds are finite pl-pseudomanifolds, that pl-pseudomanifolds have a fixed stratification (the previous one if a triangulation is given), and that all triangulations are *admissible*, i.e. compatible with the pl-structure. Our definition of pseudomanifold with smooth boundary is consistent with the definition of pseudomanifold with boundary of [9] 5.2, taking a manifold for boundary, namely assuming the singular locus of the boundary vanishes.

3.2. Intersection homology and relative intersection homology for pseudomanifolds with smooth boundary. Let first recall the basic ingredients for the definition of intersection homology, as in [9]. A *perversity* is a finite sequence of integers $\mathbf{p} = \{\mathbf{p}_j\}_{j=2}^n$ such that $\mathbf{p}_2 = 0$ and $\mathbf{p}_{j+1} = \mathbf{p}_j$ or $\mathbf{p}_j + 1$. The perversity: $\mathbf{m} = \{\mathbf{m}_j = [j/2] - 1\}$ is called *lower middle perversity*. The *null perversity* is $0_j = 0$, and the *top perversity* is $\mathbf{t}_j = j - 2$. Given a perversity \mathbf{p} , the *complementary perversity* \mathbf{p}^c is $\mathbf{p}_j^c = \mathbf{t}_j - \mathbf{p}_j = j - \mathbf{p}_j - 2$. Now let X be a pseudomanifold with boundary and with a given stratification. If j is an integer and \mathbf{p} a perversity, a pl-subspace A of X is said *(\mathbf{p}, j) -allowable* if

$$\dim(A) \leq j, \quad \dim(A \cap X_{n-k}) \leq j - k + p_k, \quad \forall k \geq 2.$$

In standard references intersection homology is usually defined for pseudomanifolds without boundary, and relative intersection homology for pairs (X, A) where A is an open subspace of a closed pseudomanifold. In order to extend the definition to pseudomanifolds with smooth boundary we have, at least, two possible equivalent approaches. The first approach is based on [18], and use smoothly enclosed subspaces, as follows. Glue the infinite cylinder $\partial X \times [0, \infty)$ to X through the boundary, let $Z = X \cup_{\partial X} \partial X \times [0, \infty)$. Embed Z into the suitable $\mathbb{R}^{k=k_1+k_2}$, in such a way that $i(\partial X) = i(X) \cap \mathbb{R}^{k_1} \times \{0, \dots, 0\}$, $i(\Sigma) \subset \{x \in \mathbb{R}^k \mid x_j > 0, \forall j > k_1\} = \mathbb{R}^{k_1} \times \mathbb{R}_+^{k_2} - \mathbb{R}^{k_1} \times \{0, \dots, 0\}$, and $i(\partial X \times [0, \infty)) \subset \{x \in \mathbb{R}^k \mid x_j \leq 0, \forall j > k_1\} = \mathbb{R}^{k_1} \times \mathbb{R}_-^{k_2}$, where i denotes the embedding. Then a Whitney stratification of \mathbb{R}^k is given by setting $Z_0 = \mathbb{R}^k - i(Z)$, $Z_1 = i(X - \Sigma)$, $Z_k = i(X_k - X_{k-1})$ for $k \geq 2$ (see [18] Section 7.1.2), and $i(X)$ is the closure of Z_1 . The subsets $S_{\pm} = \mathbb{R}^{k_1} \times \mathbb{R}_{\pm}^{k_2}$ are smoothly enclosed in \mathbb{R}^n , as in the definition of Section 1.3.2 of [18]. Now (identifying the different spaces with their images under i) it is clear that $S_+ \cap Z = X$. By definition [18] Section 1.2.3, X is smoothly enclosed in Z , and $\partial X = (S_- \cap Z) \cap (S_+ \cap Z) = (S_- \cap Z) \cap X$ is the intersection

with another smoothly enclosed subset of Z . It follows from [18] Section 1.2.3 that both the intersection chain complex $I^p C(X)$ of X , and the relative intersection chain complex $I^p C(X, \partial X)$ of the pair $(X, \partial X)$ are defined, the first as in the boundary less case, the last one by setting $I^p C_q(X, \partial X) = I^p C_q(X) / I^p C_q(\partial X)$. The intersection homology groups are the homology groups of $I^p C(X)$, and the relative intersection homology group $I^p H_q(X, \partial X)$ of the pair is defined as the q -homology group of the chain complex $I^p C(X, \partial X)$. Moreover, there is the following homology long exact sequence associated to the pair $(X, \partial X)$ (see also [10] 1.11):

$$\dots \rightarrow I^p H_q(\partial X) \rightarrow I^p H_q(X) \rightarrow I^p H_q(X, \partial X) \rightarrow I^p H_{q-1}(\partial X) \rightarrow \dots$$

The second approach proceeds as in [11] 1.4, and consists in replacing the pseudomanifold X with boundary ∂X by the pseudomanifold $X - \partial X$. For let X be a pseudomanifold with smooth boundary ∂X . Let $Col(\partial X)$ be an open collar neighborhood of ∂X . Then, $X - \partial X$ is a pseudomanifold, with open subspace $Col(\partial X) - \partial X$, and usual intersection homology theory and relative intersection homology theory are defined for $X - \partial X$, and $(X - \partial X, Col(\partial X) - \partial X)$, [11] 1.3 [16] 4.6. Since the boundary is disjoint from the singular stratum, there exists a stratum preserving homotopy self equivalence $X \sim X - \partial X$, and the same for the pair $(X, \partial X) \sim (X - \partial X, Col(\partial X) - \partial X)$, as defined in [16] 4.8. It follows by [16] 4.8.5, that $I^p H_q(X) \cong I^p H_q(X - \partial X)$, and $I^p H_q(X, \partial X) = I^p H_q(X - \partial X, Col(\partial X) - \partial X)$.

We recall now the definition of the intersection homology chain complex and groups, as in [9]. Let T be an (admissible) triangulation of X such ∂X is triangulated by a subcomplex $L = \partial T$ of T . Let $C^T(X) = C(T)$ denote the chain complex of simplicial chains of X with respect to T . Let $C(X)$ denote the direct limit chain complexes under refinement of the $C^T(X)$ over all triangulations of X compatible with the pl-structure. Since ∂X is pl-subspace of X , $C^T(\partial X) = C(L)$ is defined, and is a sub complex of $C^T(X)$, and the relative chain complex is also defined $C^T(X, \partial X) = C(T, L) = C(T)/C(L)$. The construction commutes with direct limit, and hence the $C(X)$ and $C(X, \partial X)$ are defined. The *intersection chain group* of perversity \mathbf{p} , is the subgroup $I^p C_q(X)$ of $C_q(X)$ consisting of those chains c such that $|c|$ is (\mathbf{p}, q) -allowable and $|\partial c|$ is $(\mathbf{p}, q-1)$ -allowable. The *relative intersection chain group* of perversity \mathbf{p} , is $I^p C_q(X, \partial X) = I^p C_q(X) / I^p C_q(\partial X)$, where $I^p C_q(\partial X) = C_q(\partial X)$ for each \mathbf{p} , since ∂X is actually a manifold. The group $I^p C_q(X) / I^p C_q(\partial X)$ is the subgroup of $C_q(X, \partial X)$ consisting of those chains c in T that are not in L , such that $|c|$ is (\mathbf{p}, q) -allowable and $|\partial c|$ is $(\mathbf{p}, q-1)$ -allowable.

Intersection cohomology is defined as the algebraic dual of intersection homology (see for example [16] 4.2.8). Poincaré duality is recovered for pseudomanifolds using intersection homology: with coefficients in a field, when $\partial X = \emptyset$, there is an isomorphism [9] 3.3

$$(3.1) \quad I\mathcal{P} : I^p H_q(X) \rightarrow I^{\mathbf{p}^c} H^{m-q}(X).$$

For a pseudomanifold with (smooth) boundary, the duality reads [28]

$$(3.2) \quad I\mathcal{P} : I^p H_q(X) \rightarrow I^{\mathbf{p}^c} H^{m-q}(X, \partial X).$$

3.3. Basic sets. In order to define intersection torsion and relative intersection torsion, we introduce some chain complexes of free modules. Let X be a pseudomanifold of dimension n with smooth boundary, and fixed stratification. First, we define the basic R sets as in [9] 3.4. Let T be a triangulation of X compatible with the filtration. Let $R_q^{\mathbf{p}}$ be the subcomplex of the first barycentric subdivision T' of T consisting of all simplices which are (\mathbf{p}, q) -allowable. Then, $R_q^{\mathbf{p}}$ is a subcomplex of the q -skeleton of T' . It is clear that $R_q^{\mathbf{p}}$ is a subcomplex of $R_{q+1}^{\mathbf{p}}$. Define the complex $C^{\mathbf{p}}(X)$ by setting

$$C_q^{\mathbf{p}}(X) = H_q(R_q^{\mathbf{p}}, R_{q-1}^{\mathbf{p}}),$$

and boundary defined by the homology long exact sequence of the pair (R_q^p, R_{q-1}^p) . This is a free abelian group generated by finitely many chains with contractible support. So $C_q^p(X)$ is in one one correspondence with the group of simplicial q -chains c_q with $|c_q| \subset R_q^p$, and $|\partial c_q| \subset R_{q-1}^p$. The homology of $C^p(X)$ is canonically isomorphic to $\text{Im}(H_q(R_q^p) \rightarrow H_q(R_{q+1}^p))$. By [9] 3.4, there is an isomorphism $\Psi : \text{Im}(H_q(R_q^p) \rightarrow H_q(R_{q+1}^p)) \cong I^p H_q(X)$. For this is stated, without proof, in [9] 3.4, however, if in the present case we remove the boundary, we obtain the isomorphism for the pseudomanifold $X - \partial X$, and it is clear that the groups at the two sides of the isomorphism are the same for X and $X - \partial X$, since the singular locus is disjoint from the boundary. Also note that the isomorphism is natural, that is to say is induced by the inclusion of R_q^p into T' . This is clear from the construction of the similar isomorphism called Ψ for the basic sets Q in [9] 3.2.

Let $P_q^p = R_{q+1}^p \cap L$. Then, P_q^p is an R set R_q^p of ∂X , and $\dim(R_q^p) = q - 1$. Actually, $P_q^p = L'_{(q)}$ is the q -skeleton of L' . For ∂X is a manifold and hence all the simplices of any triangulation of ∂X are allowable for any perversity. Define the chain complex $C^p(\partial X)$ as above. Then, the homology of $C^p(\partial X)$ is canonically isomorphic to $\text{Im}(H_q(P_q^p) \rightarrow H_q(P_{q+1}^p))$, and there is a natural isomorphism $\text{Im}(H_q(P_q^p) \rightarrow H_q(P_{q+1}^p)) \cong I^p H_q(\partial X) = H_q(\partial X)$, that is the restriction of Ψ .

Next, we deal with the relative chain complex. We define the complex $C^p(X, \partial X)$ by setting

$$C_q^p(X, \partial X) = H_q(R_q^p \cup L', R_{q-1}^p \cup L'),$$

and boundary defined by the homology long exact sequence of the pair $(R_q^p \cup L', R_{q-1}^p \cup L')$. This is a free abelian group generated by finitely many chains with contractible support, and is in one one correspondence with the group of the simplicial q -chains c_q with the interior of $|c_q| \subset R_q^p - (R_q^p \cap L')$, and the interior of $|\partial c_q| \subset R_{q-1}^p - (R_{q-1}^p \cap L')$. It is possible to show that the homology of $C^m(X, \partial X)$ is canonically isomorphic to $\text{Im}(i''_{q,q*} : H_q(R_q^p \cup L') \rightarrow H_q(R_{q+1}^p \cup L'))$, and that $\text{Im}(i'_{q,q*} : H_q(R_q^p \cup L') \rightarrow H_q(R_{q+1}^p \cup L'))$ is isomorphic to the relative intersection homology of the pair $(X, \partial X)$.

3.4. R torsion. In order to define intersection torsion we briefly recall the definition of the torsion of a chain complex. We follows the classical definition of Milnor [19], but with a little change of notation. Let R be a ring with the invariant dimension property, and M a finitely generated free (left) R -module. Let U be a subgroup of the group R^\times of units of R , and let $K_U(R) = K_1(R)/U$ denotes the quotient of the Whitehead group of R by the subgroup generated by the classes of the elements of U . Let $x = \{x_1, \dots, x_n\}$ and $y = \{y_1, \dots, y_n\}$ be two bases for M . We denote by (y/x) the non singular n -square matrix over R defined by the change of bases $(y_j = \sum_k (y/x)_{jk} x_k)$, and we denote by $[y/x]$ the class of (y/x) in the Whitehead group $K_U(R)$. Let

$$C : \quad C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

be a bounded chain complex of finite length m of (finite dimensional) free left R -modules. Denote by $Z_q = \ker(\partial_q : C_q \rightarrow C_{q-1})$, $B_q = \text{Im}(\partial_{q+1} : C_{q+1} \rightarrow C_q)$, and $H_q(C) = Z_q/B_q$ the homology groups of C . Assume that all the chain modules C_q have preferred bases $c_q = \{c_{q,1}, \dots, c_{q,m_q}\}$, and the homology modules $H_q(C)$ are free with preferred bases h_q . Also assuming that the boundary modules B_q are free with preferred bases or using stably free bases, we fix a set of elements $b_q = \{b_{q,1}, \dots, b_{q,n_q}\}$ of B_q such that $\partial_q(b_q)$ is a basis for B_{q-1} for each q (in other words we are choosing a lift of a basis of B_{q-1}). Then the set $\{\partial_{q+1}(b_{q+1}), h_q, b_q\}$ is a basis for C_q for each q . The *Whitehead*

torsion of C with respect to the basis $h = \{h_q\}$ is the class

$$\tau_W(C; h) = \sum_{q=0}^m (-1)^q [(\partial_{q+1}(b_{q+1}), h_q, b_q/c_q)],$$

in the Whitehead group $K_U(R)$. The definition is well posed since it is possible to show that the torsion does not depend on the bases b_q . If K is a connected finite cell complexes of dimension m , with universal covering \tilde{K} , identify the fundamental group $\pi = \pi_1(K)$ with the group of the (cellular) covering transformations of \tilde{K} , the action π makes each chain module $C_q(\tilde{K}; \mathbb{Z})$ into a free module over the group ring $\mathbb{Z}\pi$, finitely generated by the natural choice of the q -cells of K . Denote the resulting complex of free finitely generated modules over π with preferred basis (obtained by the lifts of the cells) by $C(\tilde{K}; \mathbb{Z}\pi)$. If the homology modules $H_q(X; \mathbb{Z}\pi)$ are free with preferred basis h_q , the *Whitehead torsion* of K with respect to the graded basis h is the class

$$\tau_W(K; h) = w(\tau_W(C(\tilde{K}; \mathbb{Z}\pi); h)),$$

of $K_\pi(\mathbb{Z}\pi)$. If $\rho : \pi \rightarrow \text{Aut}_R(M)$ is a representation of the fundamental group in the group of the automorphisms of some free right module M over some ring with unit R , we form the twisted complex $C(K; M_\rho)$ of free finitely generated R -modules, by setting

$$C_q(K; M_\rho) = M \otimes_\rho C_q(\tilde{K}; \mathbb{Z}\pi).$$

Fixing a basis m for M , bases for these modules (and for cycles and boundary submodules) are given by tensoring with m . Assuming that the homology modules $H_q(C(K; M_\rho))$ are free with preferred graded bases $h = \{h_q\}$, then, we define the *R torsion* $\tau_R(K; \rho, h)$ of K with respect to the representation ρ and the graded basis h to be the class of $\tau_W(C(K; M_\rho); h)$ in $\tilde{K}_1(\mathbb{Z}\text{Aut}_A(M))/\rho(\pi)$. We have

$$\tau_R(K; \rho, h) = \sum_{q=0}^m (-1)^q [\rho(\partial_{q+1}(b_{q+1}), h_q, b_q/c_q)].$$

By the same procedure we define the *relative R torsion* of the pair (K, L) , and we write $\tau_R((K, L); \rho, h) = \tau_W(C((K, L); M_\rho); h)$. In particular, if (W, g) is a compact connected oriented Riemannian manifold, and ρ is an orthogonal representation, we use the de Rham maps of section 2.2 in order to fix the basis for the homology, following Ray and Singer [22]. We define the *R torsion* of (W, g) by

$$\tau_R((W, g); \rho) = \tau_R(W; \rho, \mathcal{A}(a)),$$

where a is an orthonormal graded basis for the harmonic forms. It is possible to prove that the definition does not depend on the basis a . *Relative R torsion* for a manifold with boundary is defined accordingly.

Let X be an m -pseudomanifold with smooth boundary, let T be a triangulation of X such that the boundary ∂X of X is a subcomplex ∂T of T , and let \tilde{T} be the universal covering complex of T , and $\tilde{\partial T}$ the lift of ∂T . Let \tilde{R}_q^p be the lifts of the basic sets R_q^p to \tilde{T} , and identify the fundamental group $\pi = \pi_1(X)$ with the group of the covering transformations of \tilde{T} . Note that covering transformations are simplicial, so if we set $C_q^p(\tilde{X}) = H_q(\tilde{R}_q^p, \tilde{R}_{q-1}^p)$, the action of the group of covering transformations group makes each chain group $C_q^p(\tilde{X})$ into a free module over the group ring $\mathbb{Z}\pi$, and each of these modules is finitely generated by fixing lifts of the natural choice of the q -chains that generate $C_q^p(X)$. We obtain a complex of free finitely generated modules over $\mathbb{Z}\pi$ that we denote by $C^p(\tilde{X}; \mathbb{Z}\pi)$, with preferred basis. The same procedure applies for the relative chain

complex $C_q^p(\tilde{X}, \partial\tilde{X}) = H_q(\tilde{R}_q^p \cup \partial\tilde{T}', \tilde{R}_{q-1}^p \cup \partial\tilde{T}')$, and gives the $\mathbb{Z}\pi$ -complex $C^p((\tilde{X}, \partial\tilde{X}); \mathbb{Z}\pi)$, with preferred basis obtained by lifting the chains whose supports do not intersect the boundary.

Assuming that the homology modules $H_q(C^p(\tilde{X}; \mathbb{Z}\pi))$, $H_q(C^p((\tilde{X}, \partial\tilde{X}); \mathbb{Z}\pi))$ are $\mathbb{Z}\pi$ -free with preferred graded bases $h = \{h_q\}$, we define the *intersection Whitehead torsion* of X and the *relative intersection Whitehead torsion* of the pair $(X, \partial X)$ with respect to the graded basis h to be the classes

$$I\tau_W^p(X; h) = \tau_W(C^p(\tilde{X}; \mathbb{Z}\pi); h), \quad I\tau_W^p((X, \partial X); h) = \tau_W(C^p((\tilde{X}, \partial\tilde{X}); \mathbb{Z}\pi); h),$$

in the Whitehead group $Wh(\pi_1(X)) = K_\pi(\mathbb{Z}\pi)$, respectively. Proceeding as in the smooth case, given a representation ρ of $\pi_1(X)$ we define *intersection R torsion* $I^p\tau_R(X; \rho, h) = \tau_W(C^p(X; M_\rho); h)$ of X with respect to the representation ρ and to the graded basis h , and the *relative intersection R torsion* of the pair $(X, \partial X)$, $I^p\tau_R((X, \partial X); \rho, h) = \tau_W(C^p((X, \partial X); M_\rho); h)$. If in particular a Riemannian structure is defined on the non singular part of X , L^2 forms can be used to extend the construction of Ray and Singer and to define suitable de Rham maps from L^2 harmonic forms to intersection homology, and to fix the basis h [6] [5]. Note in particular, that the basis h fixed in this way is self dual, i.e. $IP(h_q)$ is the algebraic dual of $IP(h_{n-q})$. We use the notation $I^p\tau_R((X, g); \rho)$ and $I^p\tau_R((X, \partial X, g); \rho)$ for the resulting torsions, and we define the *intersection R torsion* of X , and the *relative intersection R torsion* of $(X, \partial X)$ with respect to the representation ρ by

$$I\tau_R((X, g); \rho) = \frac{1}{2} \left(I^m\tau_R((X, g); \rho) + I^{m^c}\tau_R((X, g); \rho) \right),$$

$$I\tau_R((X, \partial X, g); \rho) = \frac{1}{2} \left(I^m\tau_R((X, \partial X, g); \rho) + I^{m^c}\tau_R((X, \partial X, g); \rho) \right).$$

In all definitions, if X is an oriented manifold stratified with only one stratum X then we obtain the classical Whitehead torsion and the classical R torsion.

4. DUALITY THEOREMS FOR INTERSECTION R TORSION OF PSEUDOMANIFOLDS WITH SMOOTH BOUNDARY

We give in this section some duality theorems for the intersection torsion of a pseudomanifold with smooth boundary that extend the duality theorems of A. Dar [6] for the boundary less case. First, we need a lemma for the L^2 harmonics form on a cone in even dimension. The proof is analogous to the proof of the similar result given in Lemma 3.5 of [14], and will be omitted. Then, we give some formulas for the torsion, that we use to prove the final duality results.

Lemma 4.1. *Assume $\dim W = 2p$ is even. Then $(\alpha_q = \frac{1}{2}(1 + 2q - 2p))$,*

$$\mathcal{H}_{\text{abs}}^q(C_l W) = \begin{cases} \mathcal{H}^q(W), & 0 \leq q < p+1, \\ \{0\}, & p+1 \leq q \leq 2p+1; \end{cases}$$

$$\mathcal{H}_{\text{rel}}^q(C_l W) = \begin{cases} \{0\}, & 0 \leq q < p+1, \\ \{x^{2\alpha_{q-1}-1} dx \wedge \varphi^{(q-1)}, \varphi^{(q-1)} \in \mathcal{H}^{q-1}(W)\}, & p+1 \leq q \leq 2p+1. \end{cases}$$

Lemma 4.2. *Let W be a compact connected oriented manifold of dimension m without boundary. Let $\rho_0 : \pi_1(C_l W) \rightarrow O(k, \mathbb{R})$ be the trivial orthogonal representation of the fundamental group. Then,*

$$2 \log I^p\tau_R((C_l W, g_C); \rho_0) = \log \tau_R((\partial C_l W, l^2 g); \rho_0) + \log I^p\tau_R((\Sigma_l W, g_\Sigma); \rho_0) + \log \tau(\mathcal{S}_m^p),$$

where $r_q = \text{rank} H_q(W)$, and

$$\begin{aligned} \log \tau(\mathcal{S}_{2p-1}^{\mathfrak{m}}) &= \log \tau(\mathcal{S}_{2p-1}^{\mathfrak{m}^c}) = \sum_{q=0}^{p-1} (-1)^q \log \left(\frac{l}{2p-2q} \right)^{r_q}, \\ \log \tau(\mathcal{S}_{2p}^{\mathfrak{m}}) &= -\frac{1}{2} \chi(W) \log 2 + (-1)^p \log l^{\frac{r_p}{2}}, \\ \log \tau(\mathcal{S}_{2p}^{\mathfrak{m}^c}) &= -\frac{1}{2} \chi(W) \log 2 - (-1)^p \log l^{\frac{r_p}{2}}, \end{aligned}$$

Proof. Let $\Sigma_l W = (0, 2l) \times W \cup \{p_0, p_{2l}\}$ be the suspension of W , realized as the smooth gluing of two copies of $C_l W$ along the boundaries. Then, we have a short exact sequence of chain complexes:

$$(4.1) \quad 0 \rightarrow C^{\mathfrak{p}}(\partial C_l W) \rightarrow C^{\mathfrak{p}}(C_l W) \oplus C^{\mathfrak{p}}(C_l W) \rightarrow C^{\mathfrak{p}}(\Sigma_l W) \rightarrow 0.$$

A formula for the torsion of an exact sequence of complexes is given by Milnor in [19] Section 3. In the present case, we can fix the chain basis of the middle complex consistently, using the basis determined by the simplices, and hence we have the following formula

$$2 \log I^{\mathfrak{p}} \tau_{\mathbb{R}}((C_l W, g_C); \rho_0) = \log \tau_{\mathbb{R}}((W, l^2 g); \rho_0) + \log I^{\mathfrak{p}} \tau_{\mathbb{R}}((\Sigma_l W, g_{\Sigma}); \rho_0) + \log \tau(\mathcal{S}_m^{\mathfrak{p}}),$$

where the complex $\mathcal{S}_m^{\mathfrak{p}}$ is defined by the exact long homology sequence associated to the exact sequence in equation (4.1), that is the Mayer Vietoris sequence

$$\mathcal{S}_m^{\mathfrak{p}} : \quad \cdots \longrightarrow I^{\mathfrak{p}} H_q(\partial C_l W) \longrightarrow I^{\mathfrak{p}} H_q(C_l W) \oplus I^{\mathfrak{p}} H_q(C_l W) \longrightarrow I^{\mathfrak{p}} H_q(\Sigma_l W) \longrightarrow \cdots.$$

More precisely, $\mathcal{S}_{m,3q}^{\mathfrak{p}} = I^{\mathfrak{p}} H_q(W)$, $\mathcal{S}_{m,3q+1}^{\mathfrak{p}} = I^{\mathfrak{p}} H_q(C_l W) \oplus I^{\mathfrak{p}} H_q(C_l W)$ and $\mathcal{S}_{m,3q+2}^{\mathfrak{p}} = H_q(\Sigma_l W)$. In order to compute the torsion of the complex $\mathcal{S}_m^{\mathfrak{p}}$, we need the chain bases. These are the bases for the homology determined by the geometry using the de Rham maps as described in Sections 2.2 and 3.4. We proceed considering the odd and the even case independently.

Case $m = 2p - 1$. We first recall the intersection homology for the cone and the suspension with middle perversity. Since both spaces have isolated singularities, the unique value which imports of the perversity is the value $\mathfrak{m}_{2p} = \mathfrak{m}_{2p}^c = p - 1$. We have (see for example [9] 6, or [16] 4.7.2, 4.7.3)

$$I^{\mathfrak{m}} H_q(C_l W) = I^{\mathfrak{m}^c} H_q(C_l W) = \begin{cases} H_q(\partial C_l W), & q < p, \\ 0, & q \geq p, \end{cases}$$

and

$$I^{\mathfrak{m}} H_q(\Sigma_l W) = I^{\mathfrak{m}^c} H_q(\Sigma_l W) = \begin{cases} H_q(\partial C_l W), & q < p, \\ \text{Im}(H_q(W) \rightarrow H_q(\Sigma_l W)) = 0, & q = p, \\ H_q(\Sigma_l W), & q > p, \end{cases}$$

Note that, beside the homology, also the basic R set with the two middle complementary perversities coincide, by the very definition. This implies that the chain complex used in the definition of the intersection torsion for these two perversities coincide and therefore the torsions coincide. We proceed by taking $\mathfrak{p} = \mathfrak{m}$, and this will also cover the complementary case.

Next, we determine the bases for the homology. We study the two cases $q < p$ and $q \geq p$ separately. When $q < p$, consider the following part of the complex $\mathcal{S}_{2p-1}^{\mathfrak{p}}$

$$\cdots \longrightarrow I^{\mathfrak{m}} H_{q+1}(\Sigma_l W) \xrightarrow{\partial_{q+1}} I^{\mathfrak{m}} H_q(\partial C_l W) \longrightarrow I^{\mathfrak{m}} H_q(C_l W) \oplus I^{\mathfrak{m}} H_q(C_l W) \longrightarrow I^{\mathfrak{m}} H_q(\Sigma_l W) \xrightarrow{\partial_q} \cdots.$$

The geometry implies that the homomorphisms $\partial_{q+1} = \partial_q$ are null, and using the previous results, all the vector spaces are isomorphic to $V = H_q(W)$, and the sequence splits as

$$(4.2) \quad 0 \rightarrow H_q(\partial C_l W) \cong V \rightarrow I^m H_q(C_l W) \cong V \oplus I^m H_q(C_l W) \cong V \rightarrow I^m H_q(\Sigma_l W) \cong V \rightarrow 0.$$

In order to fix the homology bases, let a_q be an orthonormal base for $\mathcal{H}^q(W, g)$. Then the norm of $a_{q,j}$ with the metric $l^2 g$ is

$$\|a_{q,j}\|_{l^2 g}^2 = \int_W a_{q,j} \wedge \star_{l^2 g} a_{q,j} = l^{2p-1-2q} \int_W a_{q,j} \wedge \star_g a_{q,j} = l^{2p-1-2q} \|a_{q,j}\|_g^2 = l^{2p-1-2q}.$$

So a orthonormal base for $\mathcal{H}^q(W, l^2 g)$ is $l^{-\frac{2p-1-2q}{2}} a_q$, and applying the de Rham maps we obtain

$$\begin{aligned} \mathcal{A}_{q, l^2 g}(l^{-\frac{2p-1-2q}{2}} a_{q,j}) &= l^{-\frac{2p-1-2q}{2}} \mathcal{A}_{q, l^2 g}(a_{q,j}) = l^{-\frac{2p-1-2q}{2}} \mathcal{P}_q^{-1} \mathcal{A}^{2p-1-q} \star_{l^2 g}(a_{q,j}) \\ &= l^{-\frac{2p-1-2q}{2}} l^{2p-1-2q} \mathcal{P}_q^{-1} \mathcal{A}^{2p-1-q} \star_g(a_{q,j}) = l^{\frac{2p-1-2q}{2}} \mathcal{A}_{q,g}(a_{q,j}). \end{aligned}$$

Then the basis for $H_q(\partial C_l W)$ is $l^{\frac{2p-1-2q}{2}} \mathcal{A}_{q,g}(a_q)$. Next, consider the cone $(C_l W, g_C)$. By [14] Lemma 3.5, the constant extension of the forms in a_q gives a basis for $\mathcal{H}_{\text{abs}}^q(C_l W)$. The norm of this basis elements is

$$\|a_{q,j}\|_{g_C}^2 = \int_{C_l W} a_{q,j} \wedge \star_{g_C} a_{q,j} = \int_0^l x^{2p-1-2q} dx \int_W a_{q,j} \wedge \star_g a_{q,j} = \frac{l^{2p-2q}}{2p-2q} \|a_{q,j}\|_g^2 = \frac{l^{2p-2q}}{2p-2q}.$$

So an orthonormal base for $\mathcal{H}^q(C_l W)$ is $\left(\frac{l^{2p-2q}}{2p-2q}\right)^{-\frac{1}{2}} a_q$, and, using duality (3.2),

$$\begin{aligned} \mathcal{A}_{q, g_C} \left(\left(\frac{l^{2p-2q}}{2p-2q} \right)^{-\frac{1}{2}} a_{q,j} \right) &= \left(\frac{l^{2p-2q}}{2p-2q} \right)^{-\frac{1}{2}} \mathcal{A}_{q, g_C}(a_{q,j}) = \left(\frac{l^{2p-2q}}{2p-2q} \right)^{-\frac{1}{2}} I \mathcal{P}_q^{-1} \mathcal{A}^{2p-q} \star_{g_C}(a_{q,j}) \\ &= \left(\frac{l^{2p-2q}}{2p-2q} \right)^{-\frac{1}{2}} \left(\frac{l^{2p-2q}}{2p-2q} \right) \mathcal{P}_q^{-1} \mathcal{A}^{2p-1-q} \star_g(a_{q,j}) \\ &= \left(\frac{l^{2p-2q}}{2p-2q} \right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_{q,j}). \end{aligned}$$

This give the basis for $I^m H_q(C_l W, g_C)$: $\left(\frac{l^{2p-2q}}{2p-2q}\right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$. Repeating the same process for $\mathcal{H}^q(\Sigma_l W)$ we obtain the basis of $I^m H_q(\Sigma_l W)$: $\left(\frac{l^{2p-2q}}{p-q}\right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$. We can now compute the determinants of the change of basis in the vector spaces of the sequence in equation (4.2). At $I^m H_q(\partial C_l W)$ the determinant is 1, at $I^m H_q(C_l W) \oplus H_q(C_l W)$ is $\left(\frac{l}{2p-2q}\right)^{-\frac{r_q}{2}}$ and at $I^m H_q(\Sigma_l W)$ is $2^{-\frac{r_q}{2}}$.

We consider now the case $q \geq p$. The relevant part of the sequence \mathcal{S}_{2p-1}^m reads

$$\cdots \rightarrow 0 \rightarrow I^m H_{q+1}(\Sigma_l W) \xrightarrow{\partial_{q+1}} H_q(\partial C_l W) \rightarrow 0 \rightarrow \cdots$$

Since the de Rham maps are self dual, as observed at the end of Section 3.4, and the intersection homology with perversities \mathbf{m} and \mathbf{m}^c coincide, we can use duality (3.1) to obtain the basis for $I^{\mathbf{m}^c} H_{q+1}(\Sigma_l W)$ starting with the basis for the same space obtained when $q < p$. This gives the basis $\left(\frac{l^{2q-2p+2}}{q+1-p}\right)^{-\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$ for $I^{\mathbf{m}^c} H_{q+1}(\Sigma_l W)$. Using the basis fixed above for the other space,

the determinant of the change of basis at $H_q(\partial C_l W)$ is 1 and at $I^m H_{q+1}(\Sigma_l W)$ is $\left(\frac{l}{q+1-p}\right)^{\frac{r_q}{2}}$. Now applying the definition of Reidemeister torsion to the complex \mathcal{S}_{2p-1}^m , we obtain (where D denotes the determinant of the matrix of the change of basis)

$$\begin{aligned}
\log \tau(\mathcal{S}_{2p-1}^m) &= \sum_{q=0}^{6p} (-1)^q \log D(\mathcal{S}_{2p-1,q}^m) \\
&= \sum_{q=0}^{2p} (-1)^q \log D(I^m H_q(\Sigma_l W)) + \sum_{q=0}^{p-1} (-1)^{q+1} \log D(I^m H_q(C_l W) \oplus I^m H_q(C_l W)) \\
&= \sum_{q=0}^{p-1} (-1)^q \log D(I^m H_q(\Sigma_l W)) + \sum_{q=p+1}^{2p} (-1)^q \log D(I^m H_q(\Sigma_l W)) \\
&\quad + \sum_{q=0}^{p-1} (-1)^{q+1} \log D(I^m H_q(C_l W) \oplus I^m H_q(C_l W)) \\
&= \sum_{q=0}^{p-1} (-1)^q \log 2^{-\frac{r_q}{2}} + \sum_{q=0}^{p-1} (-1)^{q+1} \log \left(\frac{l}{2p-2q} \right)^{-\frac{r_q}{2}} + \sum_{q=p+1}^{2p} (-1)^q \log \left(\frac{l}{q-p} \right)^{\frac{r_{q-1}}{2}} \\
&= \sum_{q=0}^{p-1} (-1)^q \log 2^{-\frac{r_q}{2}} + \sum_{q=0}^{p-1} (-1)^{q+1} \log \left(\frac{l}{2p-2q} \right)^{-\frac{r_q}{2}} + \sum_{q=0}^{p-1} (-1)^q \log \left(\frac{l}{p-q} \right)^{\frac{r_q}{2}} \\
&= \sum_{q=0}^{p-1} (-1)^q \log \left(\frac{l}{2p-2q} \right)^{r_q},
\end{aligned}$$

and this completes the proof in the odd case.

Case $m = 2p$. The strategy is similar to the previous one, here however we need to distinguish the perversities \mathbf{m} and \mathbf{m}^c . For $\mathbf{m}_{2p+1} = p-1$, $\mathbf{m}_{2p+1}^c = p$, and

$$\begin{aligned}
I^m H_q(C_l W) &= \begin{cases} H_q(\partial C_l W), & q < p+1, \\ 0, & q \geq p+1, \end{cases} & I^{\mathbf{m}^c} H_q(C_l W) &= \begin{cases} H_q(\partial C_l W), & q < p, \\ 0, & q \geq p; \end{cases} \\
I^m H_q(\Sigma_l W) &= \begin{cases} H_q(\partial C_l W), & q < p+1, \\ 0, & q = p+1, \\ H_q(\Sigma_l W), & q > p+1, \end{cases} & I^{\mathbf{m}^c} H_q(\Sigma_l W) &= \begin{cases} H_q(\partial C_l W), & q < p, \\ 0, & q = p, \\ H_q(\Sigma_l W), & q > p. \end{cases}
\end{aligned}$$

As before, to determine the bases for the homology, we study the two cases $q < p+1$ and $q \geq p+1$ separately. Here we need to distinguish perversities, so consider first \mathbf{m} . When $q < p+1$, the relevant part of the complex \mathcal{S}_{2p}^m is

$$(4.3) \quad 0 \rightarrow H_q(\partial C_l W) \cong V \rightarrow I^m H_q(C_l W) \cong V \oplus I^m H_q(C_l W) \cong V \rightarrow I^m H_q(\Sigma_l W) \cong V \rightarrow 0.$$

Fixing an orthonormal base a_q for $\mathcal{H}^q(W, g)$, and computing the norm in the metric $l^2 g$, as before, an orthonormal base for $\mathcal{H}^q(W, l^2 g)$ is $l^{-\frac{2p-2q}{2}} a_q$, and applying the de Rham maps we get the basis $l^{\frac{2p-2q}{2}} \mathcal{A}_{q,g}(a_q)$ for $H_q(\partial C_l W)$. Next, consider the cone $(C_l W, g_C)$. As before, but using now Lemma 4.1, the constant extension of the forms in a_q gives a basis for $\mathcal{H}_{\text{abs}}^q(C_l W)$, the norm

is $\|a_{q,j}\|_{g_C}^2 \frac{l^{2p-2q+1}}{2p-2q+1}$, and an orthonormal base for $\mathcal{H}_{\text{abs}}^q(C_l W)$ is $\left(\frac{l^{2p-2q+1}}{2p-2q+1}\right)^{-\frac{1}{2}} a_q$. Using the duality in (3.2),

$$\mathcal{A}_{q,g_C} \left(\left(\frac{l^{2p-2q+1}}{2p-2q+1} \right)^{-\frac{1}{2}} a_{q,j} \right) = \left(\frac{l^{2p-2q+1}}{2p-2q+1} \right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_{q,j}),$$

and the basis for $I^m H_q(C_l W, g_C)$ is $\left(\frac{l^{2p-2q+1}}{2p-2q+1}\right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$. The same process for $\mathcal{H}^q(\Sigma_l W)$ gives the basis of $I^m H_q(\Sigma_l W)$: $\left(\frac{2l^{2p-2q+1}}{2p-2q+1}\right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$. We can now compute the determinants of the change of basis in the vector spaces of the sequence in equation (4.3). At $I^m H_q(\partial C_l W)$ the determinant is 1, at $I^m H_q(C_l W) \oplus H_q(C_l W)$ is $\left(\frac{l}{2p-2q+1}\right)^{-\frac{r_q}{2}}$ and at $I^m H_q(\Sigma_l W)$ is $2^{-\frac{r_q}{2}}$.

Next consider the case $q \geq p+1$. The relevant part of the sequence \mathcal{S}_{2p}^m reads

$$\cdots \longrightarrow 0 \longrightarrow I^m H_{q+1}(\Sigma_l W) \xrightarrow{\partial_{q+1}} H_q(\partial C_l W) \longrightarrow 0 \longrightarrow \cdots,$$

and using self duality as in the odd case, the basis for $I^{m^c} H_{q+1}(\Sigma_l W)$ is $\left(\frac{l^{2q-2p+1}}{2q-2p+1}\right)^{-\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$. The determinant of the change of basis at $H_q(\partial C_l W)$ is 1 and at $I^m H_{q+1}(\Sigma_l W)$ is $\left(\frac{2l}{2q-2p+1}\right)^{\frac{r_q}{2}}$. Applying the definition of Reidemeister torsion to the complex \mathcal{S}_{2p}^m , we obtain

$$\begin{aligned} \log \tau(\mathcal{S}_{2p}^m) &= \sum_{q=0}^{6p+3} (-1)^q \log D(\mathcal{S}_{2p,q}^m) \\ &= \sum_{q=0}^{2p+1} (-1)^q \log D(I^m H_q(\Sigma_l W)) + \sum_{q=0}^p (-1)^{q+1} \log D(I^m H_q(C_l W) \oplus I^m H_q(C_l W)) \\ &= \sum_{q=0}^p (-1)^q \log D(I^m H_q(\Sigma_l W)) + \sum_{q=p+2}^{2p+1} (-1)^q \log D(I^m H_q(\Sigma_l W)) \\ &\quad + \sum_{q=0}^p (-1)^{q+1} \log D(I^m H_q(C_l W) \oplus I^m H_q(C_l W)) \\ &= \sum_{q=0}^{p-1} (-1)^q \log 2^{-\frac{r_q}{2}} + \sum_{q=0}^{p-1} (-1)^{q+1} \log \left(\frac{l}{2p-2q+1} \right)^{-\frac{r_q}{2}} + \sum_{q=0}^{p-1} (-1)^q \log \left(\frac{2l}{2p-2q+1} \right)^{\frac{r_q}{2}} \\ &= \sum_{q=0}^{p-1} (-1)^{q+1} \log (2)^{r_q} + (-1)^p \log \left(\frac{l}{2} \right)^{\frac{r_p}{2}}. \end{aligned}$$

It remains to deal with the case of the complementary perversity m^c . The calculations go through exactly as in the case of perversity m , using the isomorphism with the L^2 forms, since L^2 is self dual, but with the dual boundary condition for the middle dimension, as in [5]. The shift in the dimension will give a different result in the final sum, and the result is

$$\log \tau(\mathcal{S}_{2p}^{m^c}) = \sum_{q=0}^{p-1} (-1)^{q+1} \log 2^{r_q} + (-1)^{p+1} \log (2l)^{\frac{r_p}{2}},$$

completing the proof. \square

Lemma 4.3. *Let W be a compact connected oriented manifold of dimension m without boundary. Let $\rho_0 : \pi_1(C_l W) \rightarrow O(k, \mathbb{R})$ be the trivial orthogonal representation of the fundamental group. Then,*

$$\log I^p \tau_R((C_l W, g_C); \rho_0) = \log \tau_R((\partial C_l W, l^2 g); \rho_0) + \log I^p \tau_R((C_l W, \partial C_l W), g_C); \rho_0) + \log \tau(\mathcal{T}_m^p).$$

where $r_q = \text{rank} H_q(W)$, and

$$\begin{aligned} \log \tau(\mathcal{T}_{2p-1}^m) &= \log \tau(\mathcal{T}_{2p-1}^{m^c}) = \sum_{q=0}^{p-1} (-1)^q \log \left(\frac{l}{2p-2q} \right)^{r_q}, \\ \log \tau(\mathcal{T}_{2p}^m) &= (-1)^p \log l^{\frac{r_p}{2}}, \\ \log \tau(\mathcal{T}_{2p}^{m^c}) &= (-1)^{p+1} \log l^{\frac{r_p}{2}}. \end{aligned}$$

Proof. We proceed as in the proof of Lemma 4.2. Considering the short exact sequence of chain complexes associated to the pair $(C_l W, \partial C_l W)$,

$$0 \rightarrow C^p(\partial C_l W) \rightarrow C^p(C_l W) \rightarrow C^p(C_l W, \partial C_l W) \rightarrow 0,$$

by Milnor [19] 3, we have

$$\log I^p \tau_R((C_l W, g_C); \rho_0) = \log \tau_R((\partial C_l W, l^2 g); \rho_0) + \log I^p \tau_R((C_l W, \partial C_l W), g_C); \rho_0) + \log \tau(\mathcal{T}_m^p),$$

and we need to compute the torsion of

$$\mathcal{T}_m^p : \cdots \longrightarrow I^p H_q(\partial C_l W) \longrightarrow I^p H_q(C_l W) \longrightarrow I^p H_q(C_l W, \partial C_l W) \longrightarrow \cdots,$$

where $\mathcal{T}_{m,3q}^p = I^p H_q(W)$, $\mathcal{T}_{m,3q+1}^p = I^p H_q(C_l W)$ and $\mathcal{T}_{m,3q+2}^p = H_q(C_l W, \partial C_l W)$.

Case $m = 2p - 1$. The intersection homology of $\partial C_l W$ was given in the proof of Lemma 4.2, and that of $(C_l W, \partial C_l W)$ can be computed using the sequence of the pair, we get

$$I^m H_q(C_l W, \partial C_l W) = I^{m^c} H_q(C_l W, \partial C_l W) = \begin{cases} 0, & q \leq p, \\ H_{q-1}(\partial C_l W), & q > p. \end{cases}$$

In the even case torsion and homology for the complementary perversities m and m^c coincide, so fix $p = m$. When $q < p$, consider the following part of the complex \mathcal{T}_{2p-1}^m

$$I^m H_q(C_l W, \partial C_l W) = 0 \longrightarrow I^m H_q(\partial C_l W) \longrightarrow I^m H_q(C_l W) \longrightarrow I^m H_q(C_l W, \partial C_l W) = 0.$$

Let a_q be an orthonormal base for $\mathcal{H}^q(W)$. Then, as in the proof of Lemma 4.2, a basis for $H_q(\partial C_l W)$ is $l^{\frac{2p-1-2q}{2}} \mathcal{A}_{q,g}(a_q)$, and a basis for $I^m H_q(C_l W, \partial C_l W)$ is $\left(\frac{l^{2p-2q}}{2p-2q} \right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$. The determinant of the change of basis is 1 at $I^m H_q(\partial C_l W)$, and is $\left(\frac{l}{2p-2q} \right)^{-\frac{r_q}{2}}$ at $I^m H_q(C_l W)$.

When $q \geq p$, the relevant part of the sequence \mathcal{T}_{2p-1}^m is

$$(4.4) \quad 0 \longrightarrow I^m H_{q+1}(C_l W, \partial C_l W) \longrightarrow H_q(\partial C_l W) \longrightarrow 0$$

By Lemma 3.5 of [14], a basis for harmonic forms with relative boundary conditions $\mathcal{H}_{\text{rel}}^q(C_l W)$ is $\omega_q = x^{2\alpha_{q-1}-1} dx \wedge a_{q-1}$. Their norm is

$$\|\omega_{q,j}\|_{g_C}^2 = \int_{C_l W} x^{2q-2p-1} dx \wedge a_{q-1,j} \wedge \star_g a_{q-1,j} = \int_0^l x^{2q-2p-1} dx \|a_{q-1,j}\|_g^2 = \frac{l^{2q-2p}}{2q-2p}.$$

So an orthonormal base for $\mathcal{H}_{\text{rel}}^q(C_l W)$ is $\left(\frac{l^{2q-2p}}{2q-2p}\right)^{-\frac{1}{2}} \omega_q$, using duality (3.2),

$$\begin{aligned} \mathcal{A}_{q,g_C} \left(\left(\frac{l^{2q-2p}}{2q-2p} \right)^{-\frac{1}{2}} \omega_{q,j} \right) &= \left(\frac{l^{2q-2p}}{2q-2p} \right)^{-\frac{1}{2}} \mathcal{A}_{q,g_C}(\omega_{q,j}) = \left(\frac{l^{2q-2p}}{2q-2p} \right)^{-\frac{1}{2}} I\mathcal{P}_q^{-1} \mathcal{A}_{g_C}^{2p-q} \star_{g_C} (\omega_{q,j}) \\ &= \left(\frac{l^{2q-2p}}{2q-2p} \right)^{-\frac{1}{2}} \left(\frac{l^{2q-2p}}{2q-2p} \right) \mathcal{P}_{q-1}^{-1} \mathcal{A}_g^{2p-1-(q-1)} \star_g (a_{q-1,j}) \\ &= \left(\frac{l^{2q-2p}}{2q-2p} \right)^{\frac{1}{2}} \mathcal{A}_{q-1,g}(a_{q-1,j}), \end{aligned}$$

and this gives the basis for $I^m H_q(C_l W, \partial C_l W)$. The determinants of the change of basis in (4.4) are: 1 at $H_q(\partial C_l W)$, and $\left(\frac{l}{2q-2p+2}\right)^{\frac{r_q}{2}}$ at $I^m H_{q+1}(C_l W, \partial C_l W)$. Applying the definition of Reidemeister torsion to the complex \mathcal{T}_{2p-1}^m , we obtain

$$\begin{aligned} \log \tau(\mathcal{T}_{2p-1}^m) &= \sum_{q=0}^{6p} (-1)^q \log D(\mathcal{T}_{2p-1}^m) \\ &= \sum_{q=p+1}^{2p} (-1)^q \log D(I^m H_q(C_l W, \partial C_l W)) + \sum_{q=0}^{p-1} (-1)^{q+1} \log D(I^m H_q(C_l W)) \\ &= \sum_{q=p+1}^{2p} (-1)^q \log \left(\frac{l}{2p-2q} \right)^{\frac{r_q-1}{2}} + \sum_{q=0}^{p-1} (-1)^{q+1} \log \left(\frac{l}{2q-2p} \right)^{-\frac{r_q}{2}} \\ &= \sum_{q=0}^{p-1} (-1)^q \log \left(\frac{l}{2p-2q} \right)^{r_q}, \end{aligned}$$

and this complete the proof for the odd dimensional case.

Case $m = 2p$. We have $\mathfrak{m}_{2p+1} = p-1$, $\mathfrak{m}_{2p+1}^c = p$, and (the intersection homology of the cone was given in the proof of Lemma 4.2)

$$\begin{aligned} I^m H_q(C_l W, \partial C_l W) &= \begin{cases} 0, & q \leq p+1, \\ H_{q-1}(\partial C_l W), & q > p+1, \end{cases} \\ I^{m^c} H_q(C_l W) &= \begin{cases} 0, & q \leq p, \\ H_{q-1}(\partial C_l W), & q > p. \end{cases} \end{aligned}$$

Consider first the case of perversity \mathfrak{m} . When $q < p+1$, the relevant part of the sequence \mathcal{T}_{2p}^m is

$$I^m H_q(C_l W, \partial C_l W) = 0 \longrightarrow I^m H_q(\partial C_l W) \longrightarrow I^m H_q(C_l W) \longrightarrow I^m H_q(C_l W, \partial C_l W) = 0,$$

and hence bases for all the spaces are fixed by the first part of this proof: a basis for $H_q(\partial C_l W)$ is $l^{\frac{2p-2q}{2}} \mathcal{A}_{q,g}(a_q)$, and a basis for $I^m H_q(C_l W, \partial C_l W)$ is $\left(\frac{l^{2p-2q+1}}{2p-2q+1}\right)^{\frac{1}{2}} \mathcal{A}_{q,g}(a_q)$. The determinant of the change of basis is 1 at $I^m H_q(\partial C_l W)$, and is $\left(\frac{l}{2p-2q+1}\right)^{-\frac{r_q}{2}}$ at $I^m H_q(C_l W)$.

When $q \geq p+1$, the relevant part of the sequence \mathcal{T}_{2p}^m is

$$(4.5) \quad 0 \longrightarrow I^m H_{q+1}(C_l W, \partial C_l W) \longrightarrow H_q(\partial C_l W) \longrightarrow 0$$

By Lemma 4.1, a basis for harmonic forms with relative boundary conditions $\mathcal{H}_{\text{rel}}^q(C_l W)$ is $\omega_q = x^{2\alpha_{q-1}-1} dx \wedge a_{q-1}$. Their norm is $\|\omega_{q,j}\|_{g_C}^2 = \frac{l^{2q-2p-1}}{2q-2p-1}$. So an orthonormal base for $\mathcal{H}_{\text{rel}}^q(C_l W)$ is $\left(\frac{l^{2q-2p+1}}{2q-2p+1}\right)^{-\frac{1}{2}} \omega_q$, using duality (3.2),

$$\mathcal{A}_{q,g_C} \left(\left(\frac{l^{2q-2p-1}}{2q-2p-1} \right)^{-\frac{1}{2}} \omega_{q,j} \right) = \left(\frac{l^{2q-2p-1}}{2q-2p-1} \right)^{\frac{1}{2}} \mathcal{A}_{q-1,g}(a_{q-1,j}),$$

and this gives the basis for $I^m H_q(C_l W, \partial C_l W)$. The determinants of the change of basis in (4.5) are: 1 at $H_q(\partial C_l W)$, and $\left(\frac{l}{2q-2p+1}\right)^{\frac{r_q}{2}}$ at $I^m H_{q+1}(C_l W, \partial C_l W)$. Applying the definition of Reidemeister torsion to the complex \mathcal{T}_{2p}^m , we obtain

$$\begin{aligned} \log \tau(\mathcal{T}_{2p}^m) &= \sum_{q=0}^{6p+3} (-1)^q \log D(\mathcal{H}_q^m) \\ &= \sum_{q=p+2}^{2p+1} (-1)^q \log D(I^m H_q(C_l W, \partial C_l W)) + \sum_{q=0}^p (-1)^{q+1} \log D(I^m H_q(C_l W)) \\ &= \sum_{q=0}^{p-1} (-1)^{q+1} \log \left(\frac{l}{2p-2q+1} \right)^{\frac{r_q}{2}} + \sum_{q=0}^p (-1)^q \log \left(\frac{l}{2p-2q+1} \right)^{\frac{r_q}{2}} \\ &= (-1)^p \log l^{\frac{r_p}{2}}. \end{aligned}$$

Calculation with perversity \mathbf{m}^c are similar, up to the remark at the end of the proof of Lemma 4.2, and this completes the proof. \square

Recalling the definition of intersection torsion at the end of Section 3.4, Lemmas 4.2 and 4.3 give the following duality theorems.

Proposition 4.1. *Let W be a compact connected oriented manifold of odd dimension $m = 2p - 1$ without boundary. Let $\rho_0 : \pi_1(C_l W) \rightarrow O(k, \mathbb{R})$ be the trivial orthogonal representation of the fundamental group. Then $(r_q = \text{rank } H_q(W))$,*

$$\begin{aligned} \log I\tau_{\text{R}}((C_l W, g_C); \rho_0) &= -\log I\tau_{\text{R}}((C_l W, \partial C_l W, g_C); \rho_0) \\ &= \frac{1}{2} \log \tau_{\text{R}}((\partial C_l W, l^2 g); \rho_0) + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^q \log \left(\frac{l}{2p-2q} \right)^{r_q}. \end{aligned}$$

Proof. By Lemmas 4.2 and 4.3,

$$\begin{aligned} \log I\tau_{\text{R}}((C_l W, g_C); \rho_0) &= \log I^m \tau_{\text{R}}((C_l W, g_C); \rho_0) = \log I^{m^c} \tau_{\text{R}}((C_l W, g_C); \rho_0), \\ \log I\tau_{\text{R}}((C_l W, \partial C_l W, g_C); \rho_0) &= \log I^m \tau_{\text{R}}((C_l W, \partial C_l W, g_C); \rho_0) = \log I^{m^c} \tau_{\text{R}}((C_l W, \partial C_l W, g_C); \rho_0). \end{aligned}$$

Then the statements follows once we recall that, when m is odd, by [6] 2.8,

$$\log I^{m^c} \tau_{\text{R}}((\Sigma_l W, g_{\Sigma}); \rho_0) = -\log I^m \tau_{\text{R}}((\Sigma_l W, g_{\Sigma}); \rho_0).$$

\square

Theorem 4.1. *Let W be a compact connected oriented manifold of dimension m without boundary. Let $\rho_0 : \pi_1(C_l W) \rightarrow O(k, \mathbb{R})$ be the trivial orthogonal representation of the fundamental group. Then,*

$$\begin{aligned} \log I^m \tau_R((C_l W, g_C); \rho_0) &= (-1)^m \log I^{m^c} \tau_R((C_l W, \partial C_l W, g_C); \rho_0), \\ \log I \tau_R((C_l W, g_C); \rho_0) &= (-1)^m \log I \tau_R((C_l W, \partial C_l W, g_C); \rho_0). \end{aligned}$$

Proof. The proof in the odd case follows from Proposition 4.1 and the two lemmas. Now, if $\dim W$ is even, then by [6] 2.8,

$$\log I^{m^c} \tau_R((\Sigma_l W, g_\Sigma); \rho_0) = \log I^m \tau_R((\Sigma_l W, g_\Sigma); \rho_0),$$

therefore

$$\begin{aligned} \log I^m \tau_R((C_l W, \partial C_l W, g_C); \rho_0) &= \log I^m \tau_R((C_l W, g_C); \rho_0) - \log \tau_R((\partial C_l W, l^2 g); \rho_0) - \log \tau(\mathcal{T}_{2p}^m) \\ &= \frac{1}{2} \log I^m \tau_R((\Sigma_l W, g_\Sigma); \rho_0) + \frac{1}{2} \log \tau(\mathcal{S}_{2p}^m) - \log \tau(\mathcal{T}_{2p}^m) \\ &= \frac{1}{2} \log I^{m^c} \tau_R((\Sigma_l W, g_\Sigma); \rho_0) + \frac{1}{2} \sum_{q=0}^{p-1} (-1)^{q+1} \log 2^{r_q} + \frac{(-1)^{p+1}}{2} \log (2l)^{\frac{r_p}{2}} \\ &= \log I^{m^c} \tau_R((C_l W, g_C); \rho_0). \end{aligned}$$

□

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